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Brief Paper

On the Choice of Weighting Matrices in the Minimum Variance Controller*

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Key Words—Deadbeat control; economic systems; linear systems; model reference control; optimal control; stability.

Abstract—The application of minimum variance control in general does not result in an asymptotically stable closed loop system. In this paper we show that for injective controllable linear time-invariant systems a suitable choice of weighting matrix yields a minimum variance controller which stabilizes the system. Moreover, a weighting matrix is constructed which yields a minimum variance controller which drives any initial state to zero in finite time. By means of a simulation study the influence of the choice of the weighting matrix on the controller and the closed loop behaviour is illustrated.

1. *Introduction*

IN ECONOMICS and control engineering a lot of research has been done on the design of optimal controllers and the design of controllers which stabilize the closed loop system. Usually, an optimal controller is meant to be a controller that minimizes some loss functional over a finite or infinite time horizon. In this sense the word optimal has to be interpreted in this paper too. Moreover, in the sequel we will assume that the cost functional is quadratic. For most of the infinite time optimal controllers it has been shown that they have the property that they stabilize the system, provided the weighting matrices occurring in the cost functional are positive definite, and provided that some additional conditions on the system are satisfied (e.g. stabilizability, detectability, stability of reference trajectory) (Kwakernaak and Sivan, 1972; Chow, 1975; Maybeck, 1982; Åström, 1983; Åström and Wittenmark, 1984; Engwerda, 1986). However, the performance of such a controller depends on the particular weighting matrices chosen. Since in many problems it is not clear *a priori* how these matrices should be chosen, the selection of these matrices is an important issue in the design of optimal stabilizing controllers. Especially in economics hardly any prior information about a good choice of these weighting matrices is available. This is due to the intrinsic phenomenon that exact information about the long run development of macro-economic reference and exogenous trajectories is mostly unknown. Only in the short run to some extent variables can be predicted, which make a better argumentation about the choice of weighting matrices possible. Rustem (1981) gives an algorithm how via the way of negotiation between policy-maker and control-engineer a feasible weighting matrix can be determined. Feasible in the sense that ultimately an acceptable optimal solution is attained. Particularly in economics the system is, however, subjected to noise. For that reason it is crucial that the chosen weighting matrix is such that the corresponding optimal controller stabilizes the system by a recursive application (Engwerda, 1988). This requirement is mostly overlooked in finite time optimal control problems. Therefore, we treat in this paper the question of which weighting matrices in a cost functional with a one-period planning horizon, in the literature known as the minimum variance (MV) cost criterion, are such that the corresponding optimal controller stabilizes the closed loop system by a recursive application. We show that, under the assumption that the system is controllable, there exists a whole class of stabilizing controllers and that the choice of the "best" one among this class is again not a trivial one. The paper is organized as follows. In Section 2 first the system and cost criterion are introduced. Then, a class of weighting matrices yielding a stabilizing controller is determined. Furthermore, we construct in this section, by using Luenberger's phase canonical form, a weighting matrix which leads to a deadbeat MV-controller. In Section 3 MV-control is applied to two two-dimensional macro-economic models with one and two control variables, respectively. Two control performances are compared: one with an arbitrarily chosen weighting matrix and one based on the phase canonical form. The paper ends with a conclusion section.

2. *Minimum variance control, reference stability and phase canonical forms*

We consider the following linear, finite dimensional difference equation:

$$y(k+1) = Ay(k) + Bu(k) + c(k) + v(k), \tag{1}$$

where $y(k)$ is an n-dimensional output/target vector observed in period k; $u(k)$ is an m-dimensional input/control vector with $m \le n$; $c(k)$ is a p-dimensional deterministic input, called exogenous input, and is assumed to be known at period k; and $v(k)$ is a serially uncorrelated vector with zero mean and covariance V (white noise).

We shall assume that matrix B is injective (full column rank) and that the pair (A, B) is controllable. Now consider the cost functional

$$J = E\{(y(k) - y^*(k))^{\mathrm{T}} Q (y(k) - y^*(k))\}, \quad k = 1, 2, \ldots,$$

where $E\{\cdot\}$ denotes the expectation, $y^*(k)$ is a reference value for $y(k)$ and Q is a symmetric positive definite weighting matrix.

In the sequel we shall assume that the reference trajectory is given by the first-order difference equation

$$y^*(k+1) = A^* y^*(k), \tag{2}$$

with $y^*(0)$ and A^* known.

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Subtracting equations (2) from (1) yields

$$e(k+1) = Ae(k) + Bu(k) + x(k) + v(k), \tag{3}$$

where $e(k) := y(k) - y^*(k)$ and $x(k) := (A - A^*)y^*(k) + c(k)$.

The optimal control minimizing J, subject to (3), is then obtained by straightforward differentiation and equals

$$u(k) = -(B^TQB)^{-1}B^TQ(Ae(k) + x(k)).$$

The resulting closed loop system then reads as follows:

$$\begin{aligned} e(k+1) &= M[Ae(k) + x(k)] + v(k) \\ &:= Fe(k) + Mx(k) + v(k). \end{aligned} \tag{4}$$

Here $M := \mathrm{I} - B(B^TQB)^{-1}B^TQ$.

In order to study the asymptotic behaviour of equation (4) we introduce two definitions. Let $\|\cdot\|$ denote a norm.

Definition. The reference trajectory $\{y^*(k)\}$ is said to be weakly admissible for the minimum variance control sequence $\{u(k)\}$ if there exists an $\varepsilon > 0$ and a k_0 such that $\|E\{e(k)\}\| \le \varepsilon$ for $k \ge k_0$. The reference trajectory is strongly admissible for the control sequence $\{u(k)\}$ if $\|E\{e(k)\}\| = 0$ for $k \ge k_0$.

Theorem 1. A bounded reference trajectory $y^*(\cdot)$ is weakly admissible if the exogenous input sequence $\{c(k)\}$ is bounded and F is stable, i.e. $\lim_{n\to\infty} F^n = 0$.

Proof. Equation (4) can be rewritten as

$$\begin{aligned} e(k+1) &= Fe(k) + Mx(k) + v(k) \\ &= F(Fe(k-1) + Mx(k-1) + v(k-1)) \\ &\quad + Mx(k) + v(k) \\ &= \ldots \\ &= \sum_{i=0}^{k} F^i(Mx(k-i) + v(k-i)) + F^{k+1}e(0). \end{aligned}$$

Since, due to our assumptions on $y^*(\cdot)$ and $c(\cdot)$, $x(\cdot)$ is bounded we have that

$$\begin{aligned} \|E\{e(k+1)\}\| &= \left\| \sum_{i=0}^{k} F^iMx(k-i) + F^{k+1}e(0) \right\| \\ &\le \sum_{i=0}^{k} \|F^iM\|\,\alpha + \|F^{k+1}e(0)\|, \end{aligned}$$

where $\alpha = \sup_k \|x(k)\|$. Now $\sum_{i=0}^{\infty} \|F^iM\| \le \beta < \infty$ if F is stable. From this, the stated result is immediately obtained. □

One way to find a weighting matrix such that the corresponding controller yields a stable closed loop system is described in the next proposition.

Proposition 1. Let K be any positive definite matrix, and let Q be the corresponding positive definite solution of the following Riccati equation:

$$Q = A^T\{Q - QB(B^TQB)^{-1}B^TQ\}A + K.$$

Then the feedback gain F in equation (4) is stable.

Proof. Since the pair (A, B) is controllable, B is injective and K is positive definite this result follows immediately as a special case from Proposition 3 in Engwerda (1986). In this last mentioned paper it is also proved that under these conditions the Riccati equation possesses a unique positive definite solution. This proof is similar to the one Bertsekas provides in Section 3.1 (Bertsekas, 1976) under the assumption that positive control costs are involved. That proof works also in this case since, due to our assumptions, B^TKB is positive definite. □

Note that our stabilizing weighting matrix Q is always positive definite. That this property is not a prerequisite to obtain a stabilizing MV-controller is shown for example in Silverman (1976). There it is shown that in general also semi-positive definite weighting matrices exist which give rise to a stabilizing MV-controller.

In the remainder of this section we concentrate on the construction of a best stabilizing weighting matrix. Best, in the sense that if this weighting matrix is chosen in the cost criterion the resulting controller becomes a deadbeat one with an index that is as small as possible.

To that end we transform equation (4) into its so-called phase canonical form (Luenberger, 1967).

Theorem 2. (Phase Canonical Form.) If the pair (A, B) is controllable and $\mathrm{rank}(B) = m$ then there exist non-singular transformation matrices S and T such that $\bar{A} = SAS^{-1}$ and $\bar{B} = SBT$ with

$$\bar{A} = \begin{array}{c} k_1 \\ \\ \\ k_2 \\ \\ \\ k_m \\ \\ \end{array} \left[\begin{array}{cccccccccccc} 0 & 1 & 0 & & & & & & & & & 0 \\ & & & 1 & 0 & & & & & & & 0 \\ * & * & * & * & * & & & & & * & * & * \\ 0 & & & & 0 & 1\ 0 & & & & & 0 \\ 0 & & & & & & 1 & 0 & & & & 0 \\ * & * & * & * & * & \cdots & * & * & \cdots & * & * & * \\ 0 & & & & & & & & 0 & 1 & & 0 \\ 0 & & & & & & & & & & & 1 \\ * & * & * & * & * & \cdots & * & * & \cdots & * & * & * \end{array}\right]$$

$$\bar{B} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & & & \\ 1 & 0 & & \\ 0 & & & \\ \vdots & & & \vdots \\ 0 & & & \\ 0 & 1 & 0 & \\ & & \vdots & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

where $k_1 \ge k_2 \ge \cdots \ge k_m \ge 1$ with $\sum_{i=1}^{m} k_i = n$ the controllability indices, with k_1 as "the" controllability index, and where $*$ denotes a "free" parameter.

Theorem 3. In equation (4) $F^{k_1} = 0$ if the weighting matrix in the cost functional is chosen as $Q = S^TS$.

Proof. Premultiplying equation (4) by S we have

$$Se(k+1) = SMAS^{-1}Se(k) + SMx(k) + Sv(k).$$

Defining $\bar{e}(k)$ as $Se(k)$ and choosing $Q = S^TS$ we can rewrite this equation as follows:

$$\begin{aligned} \bar{e}(k+1) &= [\mathrm{I} - \bar{B}(\bar{B}^T\bar{B})^{-1}\bar{B}^T]\bar{A}\bar{e}(k) + SMx(k) + Sv(k) \\ &:= \bar{F}\bar{e}(k) + SMx(k) + Sv(k). \end{aligned}$$

By simple calculation it can be shown that $\bar{F} = \mathrm{diag}(D_1, \ldots, D_m)$ is nilpotent with index k_1, i.e. $\bar{F}^{k_1} = 0$, where

$$D_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & & & 1 \\ 0 & & & 0 \end{bmatrix}.$$

Note that this is consistent with Wonham (1974, pp. 122–126).

Now, since $\bar{F}^k = SF^kS^{-1}$ the result follows. □

From the proof of Theorem 3 we see that the expected closed loop system is given by

$$E\{\bar{e}(k+1)\} = \bar{F}E\{\bar{e}(k)\} + SMx(k). \tag{5}$$

Evaluating equation (5) we obtain, for starting value $\bar{e}(0) = 0$, after k_1 steps

$$E\{\bar{e}(k_1+1)\} = \bar{F}^{k_1}\bar{e}(0) + \sum_{i=0}^{k_1-1} \bar{F}^iSMx(k_1 - i),$$

or in general:

$$E\{\bar{e}(k+1)\} = \sum_{i=0}^{k_1-1} \bar{F}^i SMx(k-i) \quad \text{for} \quad k \geq k_1. \qquad (6)$$

Taking norms in this equation yields

$$\|E\{\bar{e}(k+1)\}\| \leq \sum_{i=0}^{k_1-1} \|\bar{F}^i\| \, \|SMx(k-i)\| = \varepsilon(k_1). \qquad (7)$$

From equations (6) and (7) the following conclusions can be drawn.

(i) Under the condition that the sequence of exogenous inputs $\{c(k)\}$ is bounded, all bounded reference trajectories $\{y^*(k)\}$ are weakly admissible. Note that this is conformable to the statement in Theorem 1.

(ii) If $x(k)=0$ for all k, and more in particular $y^*(k)=0$ and $c(k)=0$ for all k, the MV-controller is a deadbeat controller.

(iii) In case the number of control variables is smaller than the number of target variables, the reference trajectory $\{y^*(k)\}$ is strongly admissible for $k > k_1$ if the following equation holds for all k:

$$x(k) = (A - A^*)y^*(k) + c(k) = 0.$$

Or, using (2), $y^*(k+1) = Ay^*(k) + c(k)$.

The reader interested in an exact characterization of all obtainable reference trajectories is referred to Engwerda (1988).

3. *A simulation study*

Consider the following reduced-form model:

$$\begin{bmatrix} C(k) \\ \mathrm{I}(k) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} C(k-1) \\ \mathrm{I}(k-1) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} u_1(k-1) \\ u_2(k-1) \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} x(k) + \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix}$$

where

$C(k)$ = private consumption;
$\mathrm{I}(k)$ = gross private investment;
$u_1(k)$ = governmental expenditures;
$u_2(k)$ = money supply;
$x(k)$ = (non-controllable) exogenous input;
$v^{\mathrm{T}}(k) = (v_1(k)v_2(k))$ is a white noise vector with $\mathrm{cov}\{v(k)v^{\mathrm{T}}(s)\} = V\delta_{ks}$.

All quantities are measured in billions of dollars, in quarter k.

The simulation experiments are performed on two

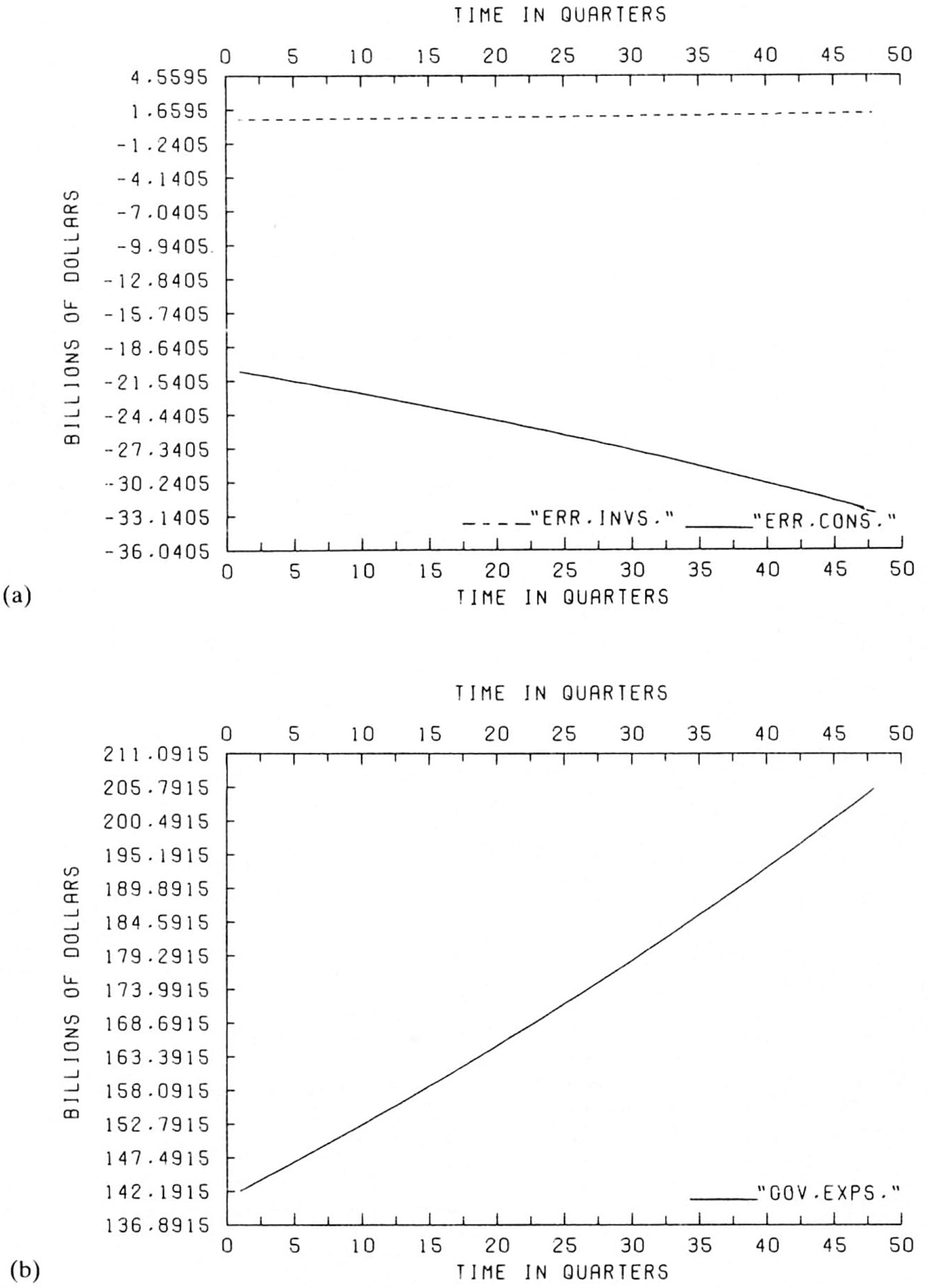


FIG. 1. $Q_1 = \mathrm{I}$, no white noise, model 1.

macro-economic models estimated by Kendrick for the U.S. economy (Kendrick, 1981, 1982). The parameters obtained by Kendrick are respectively as follows.

Model 1. An estimated macro-economic model with one control ($m = 1$).

$$A = \begin{bmatrix} 1.014 & 0.002 \\ 0.093 & 0.752 \end{bmatrix}; \quad B = \begin{bmatrix} -0.004 \\ -0.100 \end{bmatrix};$$

$$C = \begin{bmatrix} -1.312 \\ 0.448 \end{bmatrix}; \quad V = \begin{bmatrix} 9 & 0 \\ 0 & 10 \end{bmatrix}$$

with initial values: $C(0) = 460.1$; $\mathrm{I}(0) = 113.1$ and $x(0) = 10$.

Model 2. An estimated macro-economic model with two controls ($m = 2$).

$$A = \begin{bmatrix} 0.914 & -0.016 \\ 0.097 & 0.424 \end{bmatrix}; \quad B = \begin{bmatrix} 0.305 & 0.424 \\ -0.101 & 1.459 \end{bmatrix};$$

$$C = \begin{bmatrix} -0.25 \\ -0.777 \end{bmatrix}; \quad V = \begin{bmatrix} 3.73 & 0 \\ 0 & 8.58 \end{bmatrix}$$

with initial values: $C(0) = 387.9$; $\mathrm{I}(0) = 85.3$ and $x(0) = 237.75$.

To show the effect of a different choice of weighting matrix Q on the controlled system, the results of some experiments with model 1 are discussed first. We simulated with two Q matrices, namely

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q_2 = S^{\mathrm{T}}S = \begin{bmatrix} 1\,328\,874 & -44\,962 \\ -44\,962 & 1571 \end{bmatrix}.$$

The choices of these weighting matrices are motivated by the fact that $Q_1 = \mathrm{I}$ will give rise to an unstable closed loop system, whereas Q_2 makes from the minimum variance controller a deadbeat controller.

The second weighting matrix is found by taking it equal to $S^{\mathrm{T}}S$, where S is the transformation matrix obtained by transforming equation (1) into the phase canonical form (Luenberger, 1967).

The stabilization properties of a good weighting matrix are best illustrated by Figs 1a and 2b, where we assumed that all reference trajectories are generated and conformable to the system dynamics; only the initial reference values are chosen different from the initial model parameters (Engwerda, 1988). We assumed in these experiments that the model contains no white noise components. From these figures we see that the control error is, for as well consumption as

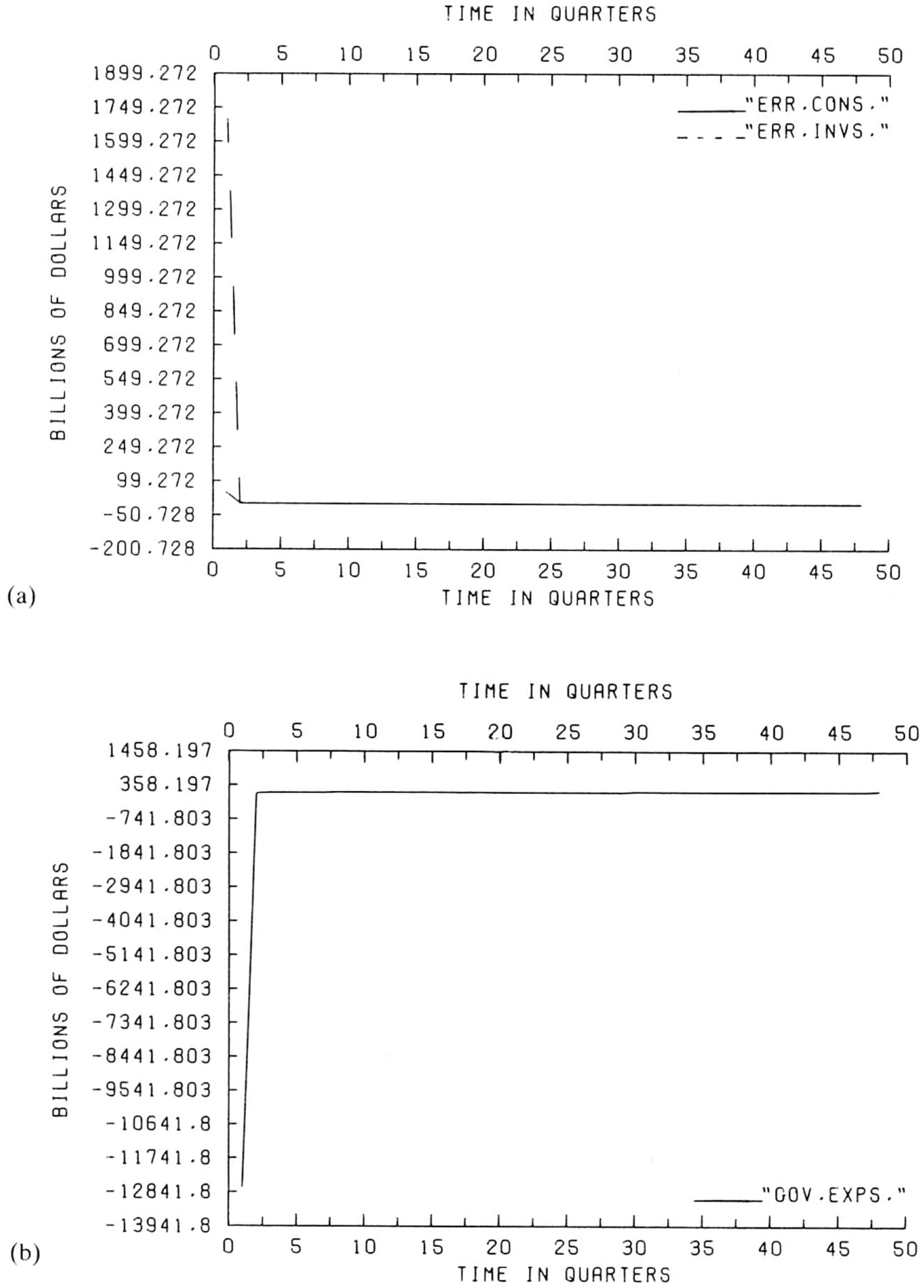
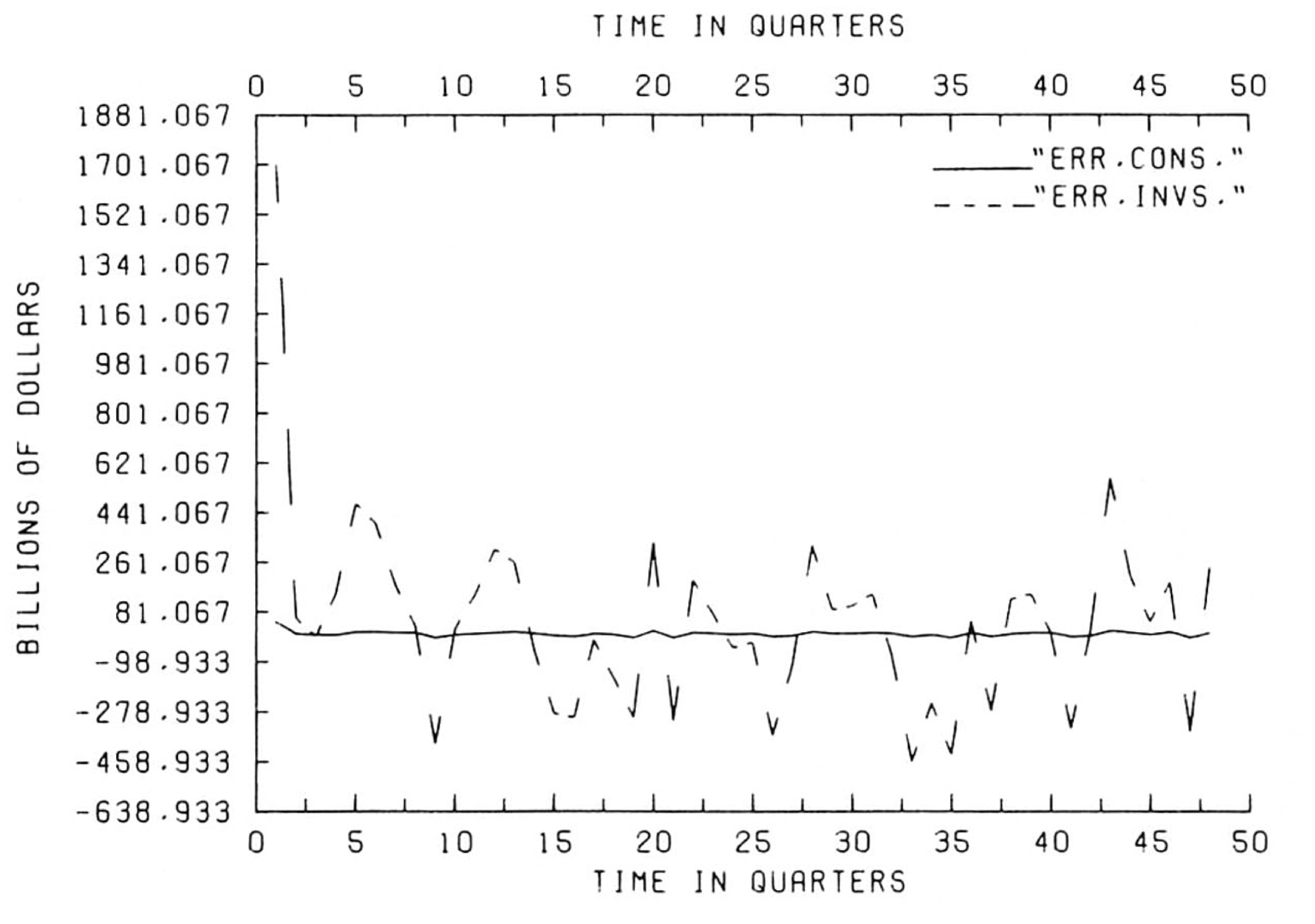
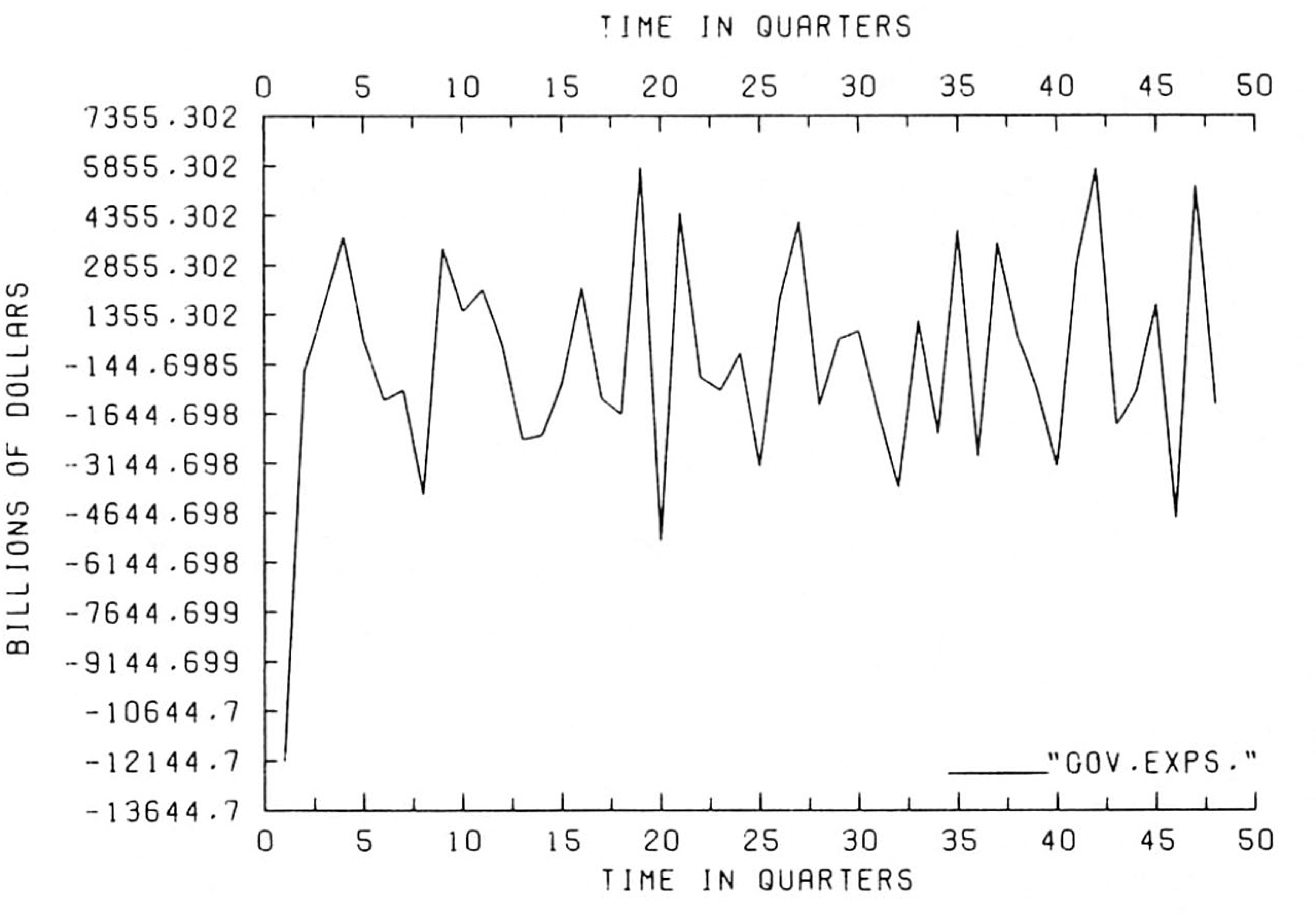


FIG. 2. $Q_2 = S^{\mathrm{T}}S$, no white noise, model 1.



(a)



(b)

FIG. 3. $Q_2 = S^{\mathrm{T}}S$, with white noise, model 1.

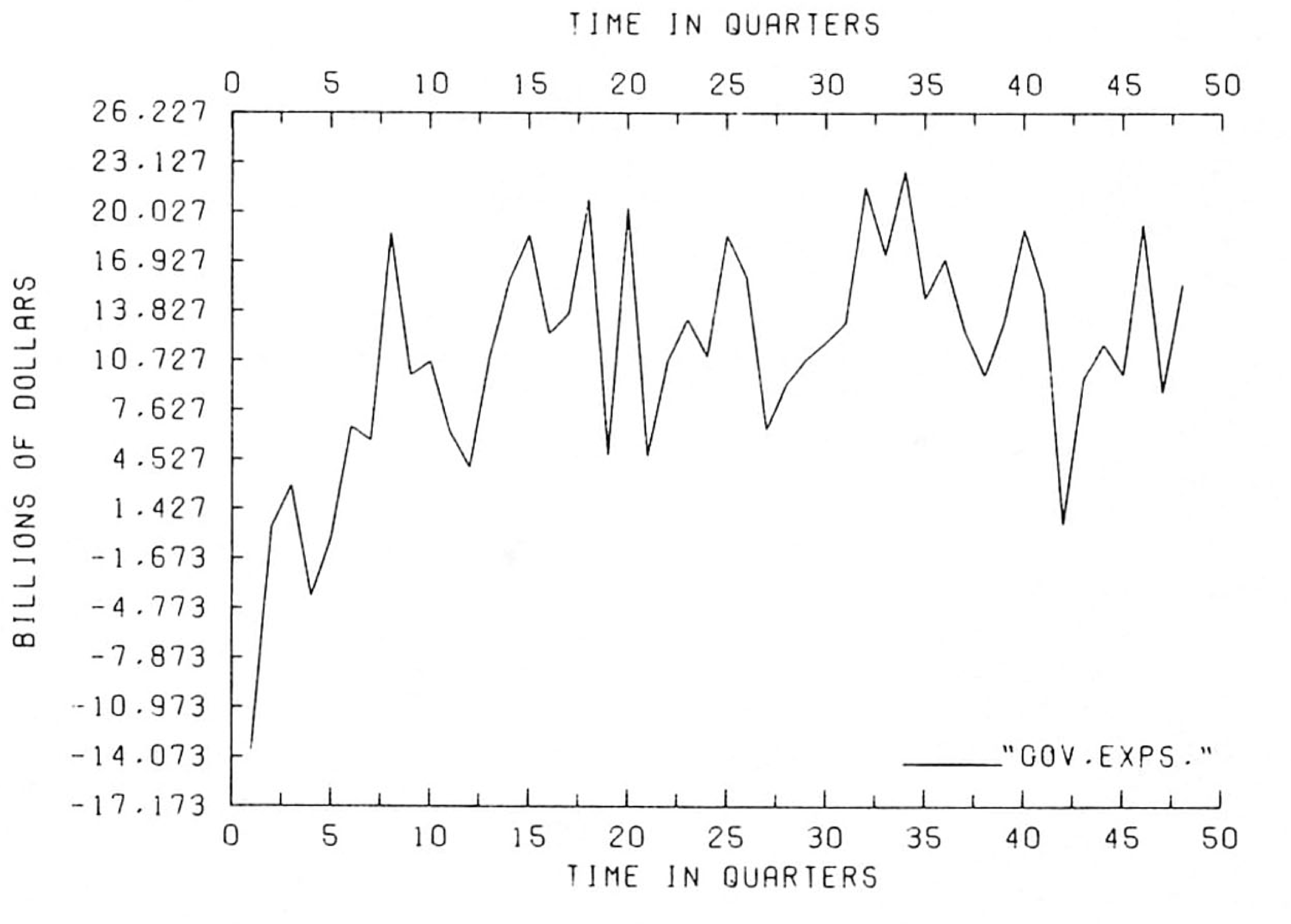


FIG. 4. $Q_3 = \mathrm{I}$, with white noise, model 2.

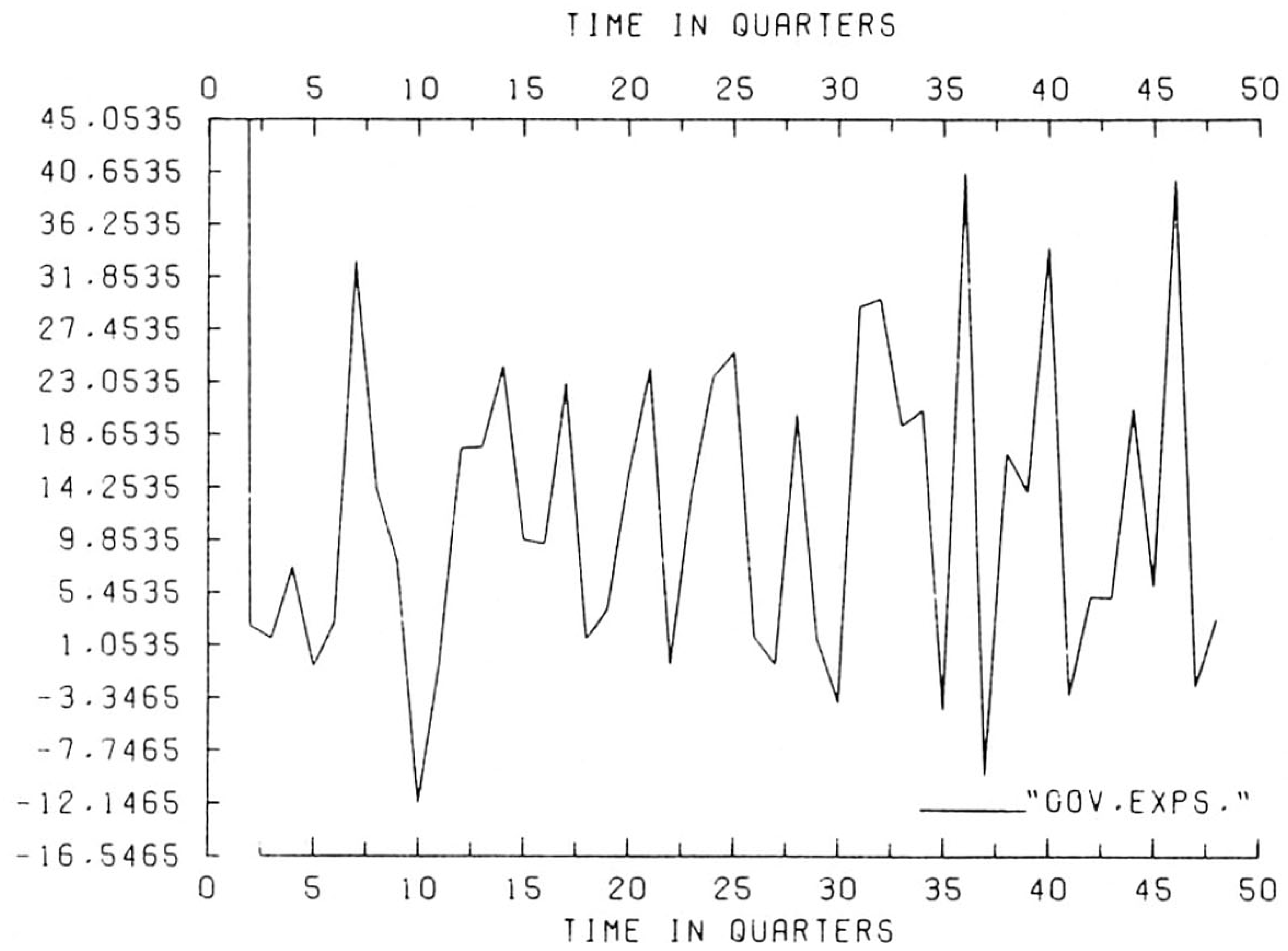


FIG. 5. $Q_4 = S^TS$, with white noise, model 2.

investment, zero from timestep 2 on the Q_2 matrix (reference stable), and steadily grows for Q_1.

In case we assume that the model does contain white noise terms we see from Figs 3b and 2b that the controller now becomes very sensitive to these terms. This is due to the big components of the Q_2 matrix.

For model 2 similar experiments were carried out for the controls $u_1(k)$ and $u_2(k)$ separately. It proved that the weighting matrix $Q = S^TS$ for obtaining a deadbeat controller in both cases was much smaller than for model 1, namely

$$Q_4 = \begin{bmatrix} 42.5 & 78.7 \\ 78.8 & 185.4 \end{bmatrix} \quad \text{respectively} \quad Q_6 = \begin{bmatrix} 60.0 & -13.9 \\ -13.9 & 3.5 \end{bmatrix}.$$

We discuss here the simulations performed with a weighting matrix $Q_3 = \mathrm{I}$ and Q_4, respectively, when the governments expenditures are used as a means of control. Again, the reference trajectories are chosen and conform to the system dynamics and the initial states of reference and model parameters different from each other. To obtain now a more realistic model, however, the sign of the exogenous terms is reversed. Since Q_3 also yields a stable closed loop system in this case, we have that the reference paths are smoothly tracked. By choosing the weighting matrix equal to Q_4 the tracking speed is increased, but this is at the expense again of a controller which is more sensitive to white noise. This is visualized in Figs 4 and 5, respectively. In this case the consequences are, however, not as dramatic as in Fig. 3b.

All other simulation results with model 2 appeared to be similar to those shown above.

4. *Conclusions*

In this paper we investigated the influence of the weighting matrix in the minimum variance control strategy on the stability of the system.

We assumed that the system is described by an injective controllable linear time-invariant recurrence equation.

We showed that, by making use of LQ theory, a whole class of weighting matrices can easily be parametrized which all give rise to a controller which stabilizes the system by a recursive application.

This parameterization may be helpful in the choice of a good weighting matrix in the MV cost criterion.

Two disadvantages of calculating a weighting matrix in this way are that the direct relationship between the designer's requirements as concerns the quality of a given process and the weighting matrix becomes less obvious, and that the simplicity of the design is lost.

An additional natural question which arises when discussing the influence of the weighting matrix on the stability property of the closed loop system is, whether there exists a weighting matrix which will make a deadbeat one from the controller. Now, it is well known that there exist direct methods to obtain a deadbeat controller. So, the existence of a weighting matrix is herewith indirectly answered.

However, an explicit relationship between deadbeat control and minimum variance control, which may help to give more insight, was lacking. We filled up this gap.

We showed that if the weighting matrix is based on the Luenberger phase canonical form a deadbeat controller is obtained.

In a simulation study we visualized the pros and cons of different choices of the weighting matrix.

First we saw that if the weighting matrix is chosen arbitrarily, the minimum variance controller does not yield in general a stable closed loop system. So the conclusion can be drawn that policy-makers must be careful in their choice of a weighting matrix when they use this type of controller to regulate the system.

The simulations show moreover that it is not self-evident that always the weighting matrix must be chosen which makes the controller a deadbeat one. The elements of this matrix may be so large that the resulting controller becomes too sensitive for small model disturbances. That is, small disturbances in the system will give rise to heavy fluctuations in the applied control. In conclusion, one can say that the choice of weighting matrices should be a well considered choice between tracking speed and disturbance sensitivity of the controller.

So the problems that emerged in the infinite time optimal control problems arise here again, extended with the problem that the chosen weighting matrix should be a stabilizing one. But now, since only a one-period ahead optimality criterion is used, there is maybe more fundamental discussion possible about the choice of weighting matrices and reference trajectories.

References

Åström, K. J. (1983). Theory and applications of adaptive control—a survey. *Automatica,* **19,** 471–486.

Åström, K. J. and B. Wittenmark (1984). *Computer Controlled Systems.* Prentice-Hall, Englewood Cliffs, NJ.

Bertsekas, D. P. (1976). *Dynamic Programming and Stochastic Control.* Academic Press, New York.

Chow, G. C. (1975). *Analysis and Control of Dynamic Economic Systems.* John Wiley, New York.

Engwerda, J. C. (1986). The solution of the infinite trajectory tracking problem for discrete time systems possessing an exogenous component. Memorandum Cosor 86-08, Eindhoven University of Technology.

Engwerda, J. C. (1988). On the set of obtainable reference trajectories using minimum variance control. *J. Econ*, **3**, 279–301.
Kendrick, D. A. (1981). *Stochastic Control for Economic Models*. McGraw-Hill, New York.
Kendrick, D. A. (1982). Caution and probing in a macroeconomic model. *J. Econ. Dyn. Control*, **4**, 149–170.
Kwakernaak, H. and D. Sivan (1972). *Linear Optimal Control Systems*. John Wiley, New York.
Luenberger, D. G. (1967). Canonical forms for linear multivariable systems. *IEEE Trans. Aut. Control*, **12**, 290–293.
Maybeck, P. S. (1982). Stochastic models, estimation, and control. In *Mathematics in Science and Engineering*, Vols 141–143. Academic Press, London.
Otter, P. W. (1985). Dynamic feature space modelling, filtering and self-tuning control of stochastic systems. In *Lecture Notes in Economics and Mathematical Systems*, Vol. 246. Springer, Berlin.
Rustem, B. (1981). *Projection Methods in Constrained Optimisation and Applications to Optimal Policy Decisions*. Springer, Berlin.
Silverman, L. M. (1976). Discrete Riccati equations: alternative algorithms, asymptotic properties, and system theory interpretations. In *Control and Dynamic Systems*, Vol. 12. Academic Press, New York.
Wonham, W. M. (1974). *Linear Multivariable Control: a Geometric Approach*. Springer, Berlin.